Lesson 14. Improving Search: Convexity and Optimality

1 Overview

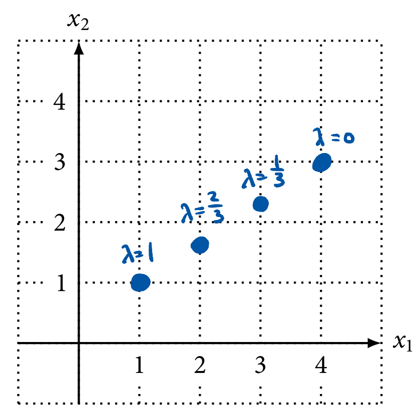
1: Find an initial feasible solution $\mathbf{x}^0$
2: Set $k = 0$
3: **while** $\mathbf{x}^k$ is not locally optimal **do**
4: Determine a new feasible solution $\mathbf{x}^{k+1}$ that improves the objective value at $\mathbf{x}^k$
5: Set $k = k + 1$
6: **end while**

- Step 3 – Improving search converges to local optimal solutions, which aren't necessarily globally optimal
- Wishful thinking: when are all local optimal solutions are in fact globally optimal?

2 Convex sets

Example 1. Let $\mathbf{x} = (1, 1)$ and $\mathbf{y} = (4, 3)$. Compute and plot $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ for $\lambda \in \{0, 1/3, 2/3, 1\}$.

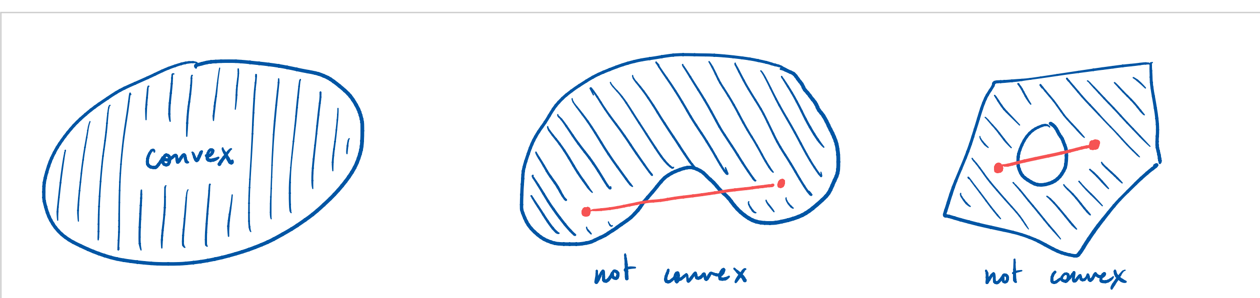
λ	$\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$
0	$0\begin{pmatrix}1\\1\end{pmatrix} + (1-0)\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}4\\3\end{pmatrix}$
1/3	$\frac{1}{3}\begin{pmatrix}1\\1\end{pmatrix} + \frac{2}{3}\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}3\\7/3\end{pmatrix}$
2/3	$\frac{2}{3}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{3}\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}2\\5/3\end{pmatrix}$
1	$1\begin{pmatrix}1\\1\end{pmatrix} + 0\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$



- Given two solutions $\mathbf{x}$ and $\mathbf{y}$, the **line segment** joining them is

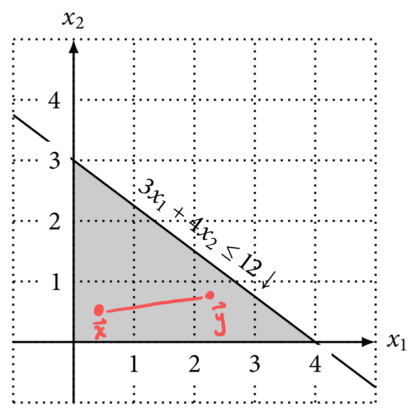
$$\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \quad \text{for } \lambda \in [0, 1]$$

- A feasible region S is **convex** if for all $\mathbf{x}, \mathbf{y} \in S$, then $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in S$ for all $\lambda \in [0, 1]$
  - A feasible region is convex if for any two solutions in the region, all solutions on the line segment joining these solutions are also in the region
- Graphically: convex vs. nonconvex



Example 2. Show that the feasible region of the LP below is convex.

$$\begin{array}{lll} \text{minimize} & 3x_1 + x_2 & \\ \text{subject to} & 3x_1 + 4x_2 \leq 12 & (1) \\ & x_1 \geq 0 & (2) \\ & x_2 \geq 0 & (3) \end{array}$$



Proof.

- Let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ be arbitrary points in the feasible region
- In other words, $\mathbf{x}$ and $\mathbf{y}$ satisfy (1), (2), (3)
- We need to show $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ also satisfies (1), (2), (3) for any $\lambda \in [0, 1]$
- Note that

$$\lambda\mathbf{x} + (1-\lambda)\mathbf{y} = \boxed{\lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (1-\lambda)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \lambda x_1 + (1-\lambda)y_1 \\ \lambda x_2 + (1-\lambda)y_2 \end{pmatrix}}$$

- One constraint at a time: does $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ satisfy (1)?

$$\begin{aligned} 3(\lambda x_1 + (1-\lambda)y_1) + 4(\lambda x_2 + (1-\lambda)y_2) &= \lambda\underbrace{(3x_1 + 4x_2)}_{\leq 12} + (1-\lambda)\underbrace{(3y_1 + 4y_2)}_{\leq 12} \\ &\leq 12\lambda + 12(1-\lambda) \\ &= 12 \qquad \Rightarrow \lambda\vec{x} + (1-\lambda)\vec{y} \text{ satisfies } (1). \end{aligned}$$

- We can show $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ also satisfies (2) and (3) in a similar fashion ☐

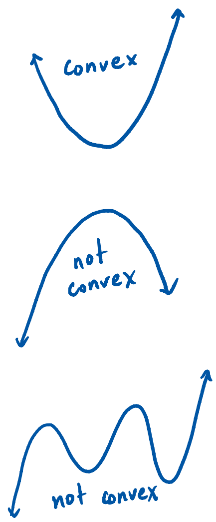
- **In general, the feasible region of an LP is convex**

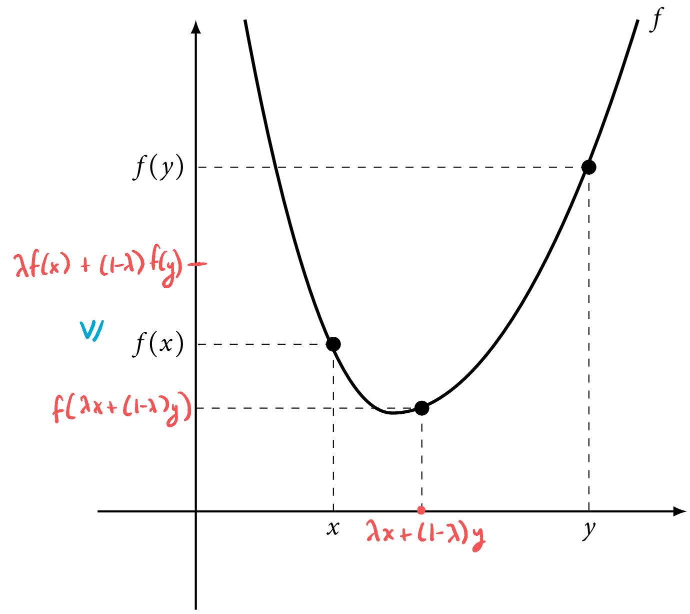
3 Convex functions

- Given a convex feasible region S, a function $f(\mathbf{x})$ is **convex** if for all solutions $\mathbf{x}, \mathbf{y} \in S$ and for all $\lambda \in [0,1]$

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y})$$

- Graphically:





Example 3. Show that the objective function of the LP in Example 2 is convex.

Proof.

- Let $f(\mathbf{x}) = 3x_1 + x_2$
- For any $\mathbf{x}$ and $\mathbf{y}$, we have:

$$\begin{aligned} f\big(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}\big) &= 3\big(\lambda x_1 + (1-\lambda)y_1\big) + \big(\lambda x_2 + (1-\lambda)y_2\big) \\ &= \lambda\underbrace{\big(3x_1 + x_2\big)}_{f(\vec{x})} + (1-\lambda)\underbrace{\big(3y_1 + y_2\big)}_{f(\vec{y})} \\ &= \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}) \qquad \square \end{aligned}$$

$$\lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (1-\lambda)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \lambda x_1 + (1-\lambda)y_1 \\ \lambda x_2 + (1-\lambda)y_2 \end{pmatrix}$$

- **In general, the objective function of an LP – a linear function – is convex**

4 Minimizing convex functions over convex sets

Big Theorem. Consider the following optimization model:

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & g_i(\mathbf{x}) \left\{ \begin{matrix} \leq \\ \geq \\ = \end{matrix} \right\} b_i \quad \text{for } i \in \{1, \ldots, m\} \end{aligned} \tag{$*$}$$

Suppose f is convex and the feasible region is convex. If $\mathbf{x}$ is a local optimal solution, then $\mathbf{x}$ is a global optimal solution.

Proof.

- By contradiction – suppose $\mathbf{x}$ is not a global optimal solution
- Then there must be another feasible solution $\mathbf{y}$ such that $f(\mathbf{y}) < f(\mathbf{x})$
- Take $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ really close to $\mathbf{x}$ (λ really close to 1)
- Since the feasible region is convex, $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ is also in the feasible region
- We have that:

$$\begin{aligned} f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) &\leq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}) && (\text{since } f \text{ is convex}) \\ &< \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{x}) && (\text{since } f(\mathbf{y}) < f(\mathbf{x})) \\ &= f(\mathbf{x}) \end{aligned}$$

- Therefore: $f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) < f(\mathbf{x})$
- $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ is a feasible solution in the neighborhood of $\mathbf{x}$ with better objective value than $\mathbf{x}$
- This contradicts $\mathbf{x}$ being a local optimal solution!
- Therefore, $\mathbf{x}$ must be a global optimal solution □

- Remember that an improving search algorithm finds local optimal solutions
- Since the objective function of an LP is convex, and the feasible region of an LP is convex:

Big Corollary 1. A global optimal solution of a minimizing linear program can be found with an improving search algorithm.

- A similar theorem and corollary exists when maximizing concave functions over convex sets
  - See pages 222–225 in Rader for details

Big Corollary 2. A global optimal solution of a maximizing linear program can be found with an improving search algorithm.